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SELF-SUSTAINED OSCILLATIONS OF RAYLEIGH AND VAN DER POL OSCILLATORS WITH MODERATELY LARGE FEEDBACK FACTORS†

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Periodic motions of essentially non-linear self-sustained oscillatory systems, described by Rayleigh and Van der Pol equations, are constructed and investigated. The period and initial value of the velocity of the system, which determine the self-sustained oscillations of the oscillators for small and moderately large values of the feedback factors, are calculated by the Lyapunov–Poincaré method using a developed accelerated-convergence algorithm and a continuation with respect to a parameter. The trajectories and limit cycles are also constructed with a guaranteed relative and absolute error. The qualitative features of the self-sustained oscillations due to an increase in the self-excitation factors are established and the oscillators are compared. The results of a numerical investigation of periodic solutions of the Van der Pol equation are compared with familiar solutions.

There is an extensive literature devoted to qualitative, analytical and numerical methods of investigating self-sustained oscillations for systems with one degree of freedom (see, for example, the monographs [1–7] and the bibliographies they contain). The qualitative and topological approaches to investigating dynamical systems in the phase plane which have been developed, and which are described by Liénard-type equations, give criteria (sufficient conditions) for the existence and stability of limit cycles (self-sustained oscillations). The position and shape of a limit cycle can be determined approximately in the phase plane using the method of isoclines [2–7] and by employing numerical methods. This approach is not effective at present for highly accurate mass operative calculations in the parametric synthesis of self-sustained oscillatory systems.

For small values of the feedback factors (the self-excitation factors), i.e. for quasi-linear self-sustained oscillatory systems, approximate analytical methods of non-linear mechanics due to Lyapunov–Poincaré, Krylov–Bogolyubov, etc. [2, 5–10] are widely used. If this factor is asymptotically large, a singularly perturbed self-sustained oscillatory system is obtained, which performs relaxation oscillations [1]. The limit cycle is then constructed approximately using Dorodnitsyn's method [11, 5] and relaxation-oscillation methods [7, 12, 13].

The intermediate region of variation of the self-excitation coefficients is of considerable interest from the theoretical, procedural, computation and particularly the applied aspects. There is an extremely limited number of publications, the results of which relate to constructing periodic modes of operation (limit cycles) for self-sustained oscillatory systems with moderately large values of the dimensionless feedback factors. We note the numerical results obtained for the Van der Pol equation in [14, 15] without justifying the corresponding estimates of the accuracy of the calculations. The approaches employed are extremely cumbersome and cannot easily be generalized to self-sustained oscillatory systems of large dimensions.

The numerical–analytical method of accelerated convergence in combination with the continuation with respect to a parameter (self-excitation factor) developed in [16] enables one, using the Lyapunov–Poincaré method [10], to construct the required solutions for a wide range of self-sustained oscillatory systems. In particular, the case of the Van der Pol type equation with a non-linear restoring force, proportional to the third and fifth powers of the deviation, has been investigated. The dependence of the periods and the amplitudes of the self-sustained oscillations on the feedback factor was established

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in [16]. Below, using this approach we obtain a practically exact solution (with a relative error of 10^{-6}–10^{-8}) of the problems of the self-sustained oscillations of Rayleigh and Van der Pol oscillators and we compare their main characteristics, namely, the periods, amplitudes, phase trajectories and limit cycles.

1. REDUCTION OF THE EQUATIONS TO STANDARD FORM

Consider a Liénard-type equation containing power-law functions of the generalized coordinate x and the velocity $\dot{x}$ of the form

$$\begin{gathered} m\ddot{x} + g(x,\dot{x})\dot{x} + f(x) = 0 \\ g = -k|x|^{\beta}|\dot{x}|^{\gamma-1} + l|x|^{\delta}|\dot{x}|^{\sigma-1}, \quad f = c|x|^{\alpha-1}x \\ m, c, k, l > 0; \quad \infty > \alpha, \beta, \gamma, \delta, \sigma \geq 0; \quad \delta + \sigma > \beta + \gamma \end{gathered} \tag{1.1}$$

The constant m has the meaning of an inertial characteristic, while c is the coefficient of elasticity (non-linear when $\alpha \neq 1$) of the restoring force f. If $0 < \alpha \ll 1$, the function f is close to a Rayleigh function when $|x| \sim 1$; when $\alpha = 0$ we have $f = c\,\mathrm{sign}\,x$, $x \neq 0$.

It follows from the qualitative theory of dynamical systems [2–5] that, for certain sufficient conditions, the stationary point $x = \dot{x} = 0$ is unstable. Equation (1.1) allows of a periodic solution, to which the stable limit cycle corresponds. Since the structure of Eq. (1.1) does not change when we make the change of variables $x \to -x, \dot{x} \to -\dot{x}$, the limit cycle is a centrally symmetric closed curve, where $x(t + T) \equiv -x(t)$, and $2T$ is the period of the self-sustained oscillations. Hence, it is sufficient to obtain the required functions in any half-period. For specific values of the system parameters the required solution and its characteristics (the period, amplitude, etc.) can be constructed numerically (see below).

Equation (1.1) contains four dimensional parameters m, c, k and l. We introduce the following dimensionless quantities by linear transformations of the variable x and the argument t

$$\begin{gathered} x^* = d^{-1}x, \quad d = (k/l)^{(1-\kappa)/\xi}(c/m)^{-\eta/(\lambda\xi)} \\ t^* = \nu t, \quad \nu = (k/l)^{(\alpha-1)/(\lambda\xi)}(c/m)^{1/\lambda} \\ \xi = \delta + \sigma - \beta - \gamma > 0, \quad \eta = \sigma - \gamma, \quad \lambda = 2 + (\alpha - 1)\eta/\xi > 0 \\ \kappa = (\alpha - 1)\eta/(\lambda\xi) = (\alpha - 1)\eta/[2\xi + (\alpha - 1)\eta] \neq 1 \end{gathered} \tag{1.2}$$

Further, Eq. (1.1) will be rewritten in dimensionless variables x^* and t^* (1.2) (for brevity the asterisk will henceforth be omitted). It contains a single dimensionless parameter $\varepsilon > 0$ – the feedback factor or the self-excitation factor of the self-sustained oscillations

$$\begin{gathered} \ddot{x} + \varepsilon(-|x|^{\beta}|\dot{x}|^{\gamma-1} + |x|^{\delta}|\dot{x}|^{\sigma-1})\dot{x} + |x|^{\alpha-1}x = 0 \\ \varepsilon = \frac{k}{m}d^{\beta+\gamma-1}\nu^{\gamma-2} = \frac{l}{m}d^{\delta+\sigma-1}\nu^{\sigma-2} \equiv m^M c^C l^L k^K \\ M = -1 - C, \quad C = \frac{1}{\lambda\xi}(\gamma\delta - \sigma\beta + 2\beta + \gamma - 2\delta + \sigma) \\ L = \frac{1}{\lambda\xi}(\lambda(\kappa - 1)(\beta + \gamma - 1) + (1 - \alpha)(\gamma - 2)) \\ K = \frac{1}{\lambda\xi}(-\lambda(\kappa - 1)(\delta + \sigma - 1) + (1 - \alpha)(\sigma - 2)) \end{gathered} \tag{1.3}$$

The parameter ε in (1.3) is expressed in terms of the initial dimensional quantities m, c, l and k (1.1) and is a lengthy power-law function with rather complex expressions for the exponents M, C, L and K in terms of the initial exponents α, β, γ, δ, σ. With the above assumptions, the numerical values of M, C, L and K are bounded, since $\lambda\xi > 0$ by relations (1.2). In particular, for a Van der Pol type equation with linear and non-linear functions $f(x)$ (1.1) of the restoring force, by means of the change of variables (1.2) we obtain the following representation [16]

$$\begin{gathered}
\ddot{x} - \varepsilon(1 - x^2)\dot{x} + |x|^{\alpha - 1}x = 0, \quad \alpha \geq 0 \\
\beta = 0, \quad \gamma = \sigma = 1, \quad \delta = 2; \quad d = (k/l)^{1/2}, \quad \nu = (c/m)^{1/2}(k/l)^{\chi} \\
\chi = (1 - \alpha)/4, \quad \varepsilon = k(cm)^{-1/2}(k/l)^{-\chi}
\end{gathered} \tag{1.4}$$

When $\alpha = 1$, i.e. in the case of the classical Van der Pol equation, we have $\chi = 0$, $\varepsilon = k/\sqrt{cm}$; the self-excitation factor is independent of the parameter l. Self-sustained oscillations described by Eq. (1.4) have been investigated in detail in [16] for $\alpha = 3$ and 5.

In applied problems, Van der Pol type equations are usually obtained using artificial methods from the initial ones, which have the form of Rayleigh equations and which describe different physical processes of a self-sustained oscillatory character [1, 5–8, 10]. For a Rayleigh type equation, similar to (1.4) we obtain the representation

$$\begin{gathered}
\ddot{x} - \varepsilon(1 - \dot{x}^2)\dot{x} + |x|^{\alpha - 1}x = 0, \quad \alpha \geq 0 \\
\beta = \delta = 0, \quad \gamma = 1, \quad \sigma = 3; \quad d = (km/cl)^{\rho}, \quad \nu = (m/c)^{-\rho}(k/l)^{1/2(\alpha - 1)} \\
\rho = 1/(\alpha + 1), \quad \varepsilon = (d/m)\sqrt{kl}
\end{gathered} \tag{1.5}$$

In a similar way, for the classical Rayleigh equation (with $\alpha = 1$), we obtain expressions for the normalizing coefficients: $\nu = (c/m)^{1/2}$, $d = (km/cl)^{1/2}$; the self-excitation factor $\varepsilon = (k/mc)^{1/2}$ is also independent of the parameter l. Note that all the main characteristics of the self-sustained oscillations (apart from the period) will depend on l in the case of the classical Rayleigh and Van der Pol equations.

2. SELF-SUSTAINED OSCILLATIONS OF A RAYLEIGH OSCILLATOR FOR MODERATELY LARGE SELF-EXCITATION FACTORS

We will consider the problem of a highly accurate construction of the limit cycles and trajectories for Rayleigh's equations for moderately large values of the parameter of the problem

$$\begin{gathered}
\ddot{x} - \varepsilon(1 - \dot{x}^2)\dot{x} + x = 0, \quad 0 \leq \varepsilon \leq \varepsilon^* \quad (\varepsilon \sim 10) \\
x(0) = x(2T), \quad \dot{x}(0) = \dot{x}(2T)
\end{gathered} \tag{2.1}$$

The half-period T is unknown and is to be determined together with the other characteristics of the self-sustained oscillations. The unique stable periodic solution is constructed using the Lyapunov–Poincaré method [10] in the same way as described previously in [16]. We replace the argument t by τ with the aim of clearly distinguishing the dependence on the unknown T. In view of the central symmetry, as pointed out in Section 1, it is sufficient to confine ourselves to considering the problem over a half-period $\Delta\tau = \Theta$, where $\Theta > 0$ is any fixed number, it is convenient to put $\Theta = 1$. As a result we obtain the boundary-value problem

$$\begin{gathered}
\ddot{x} - \varepsilon T(1 - T^{-2}\dot{x}^2)\dot{x} + T^2 x = 0, \quad x = x(\tau, \varepsilon) \\
x(0, \varepsilon) = 0, \quad \dot{x}(0, \varepsilon) = b, \quad x(1, \varepsilon) = 0, \quad \dot{x}(1, \varepsilon) = -b \\
T = T(\varepsilon), \quad b = b(\varepsilon), \quad 0 \leq \tau \leq 1
\end{gathered} \tag{2.2}$$

Here and henceforth the dots again denote derivatives with respect to the argument τ. The problem contains four unknown parameters (two constants of integration and the parameters T and b), which are found from the four boundary conditions. Its solution enables us to obtain the "right half" of the limit cycle. The "left half" is centrally symmetric; it is obtained by replacing $\tau = 0$ by $\tau = 2$ or $\tau = -1$.

The boundary-value problem for constructing a solution corresponding to the "upper" (or "lower") half of the cycle [16] is formulated in a similar way,

$$\begin{gathered}
x(0, \varepsilon) = -a, \quad \dot{x}(0, \varepsilon) = 0; \quad x(1, \varepsilon) = a, \quad \dot{x}(1, \varepsilon) = 0 \\
(x = -a, \dot{x} = 0, \tau = 2, -1), \quad a = a(\varepsilon)
\end{gathered} \tag{2.3}$$

The unknown a in problem (2.3) has the mechanical meaning of the amplitude of the self-sustained oscillations. Its determination (in addition to b and T) and its investigation as a function of the parameter ε is of considerable interest when investigating the self-sustained oscillations of a Rayleigh oscillator.

We will describe the procedure for the numerical-analytical solution of problem (2.2) very briefly. We represent the equation in the standard Cauchy form by introducing the variable velocity $y = \dot{x}$. In addition, we introduce sensitivity functions (p, w) and (q, z), which are derivatives of the solution (x, y) with respect to the parameters T and b; we obtain the following relations (the dependence of the unknown functions and parameters on ε is not indicated for brevity)

$$\begin{aligned}
&\dot{x} = y, \quad \dot{y} = -T^2x + \varepsilon T(1 - T^{-2}y^2)y; \quad x(0) = x(1) = 0, \quad y(0) = b, \quad y(1) = -b \\
&\dot{p} = q, \quad \dot{q} = -T^2p + \varepsilon T(1 - 3T^{-2}y^2)q, \quad p = \partial x/\partial b; \quad p(0) = 0, \quad q(0) = 1 \\
&\dot{w} = z, \quad \dot{z} = -T^2w - 2Tx + \varepsilon T(1 - 3T^{-2}y^2)z + (1 + T^{-2}y^2)y \\
&w = \partial x/\partial T, \quad z = \partial y/\partial T; \quad w(0) = z(0) = 0
\end{aligned} \tag{2.4}$$

The boundary-value problem for x, y is formally independent of the unknowns p, q, w and z. After determining $x(\tau)$, $y(\tau)$ and T these functions are found by integrating two second-order independent linear Cauchy problems (see (2.4)).

However, the sensitivity functions (p, w) and (q, z) introduced above, i.e. their values when $t = 1$, enable us to refine the deficient values of the parameters T and b (according to boundary conditions (2.3)) in an accelerated-convergence Newton-type accelerated-convergence on the basis of certain estimates $T_0(\varepsilon)$ and $b_0(\varepsilon)$. For sufficiently small $\varepsilon > 0$ we can take values corresponding to the zeroth approximation in the perturbation method [10] $T_0(0) = \pi$, $b(0) = 2\pi/\sqrt{3}$, $a(0) = 2/\sqrt{3}$. By successively increasing the parameter ε in conjunction with extrapolation of the quantities $T(\varepsilon)$, $b(\varepsilon)$ and $a(\varepsilon)$ by means of the rapidly converging method [16, 17], by highly accurate integration of the Cauchy problems (2.4) we can construct periodic functions $x(t, \varepsilon)$, $\dot{x}(t, \varepsilon)$ and the required quantities $T(\varepsilon)$, $b(\varepsilon)$ and $a(\varepsilon)$ with the required relative and absolute accuracy for moderately large values of ε: $0 < \varepsilon \leqslant \varepsilon_0 \sim 10\text{–}10^2$.

The main results of the calculations consist in finding the required quantities $T(\varepsilon)$ and $b(\varepsilon)$ for $0 \leqslant \varepsilon \leqslant 10$; they are presented below:

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.8
T	3.14356	3.14946	3.15990	3.17287	3.19033	3.21156	3.26414
b	3.62775	3.62926	3.63169	3.63519	3.63996	3.64630	3.66447
ε	1.0	1.2	1.4	1.6	1.8	2.0	2.2
T	3.33164	3.41061	3.50007	3.59827	3.70368	3.81493	3.93099
b	3.69180	3.72997	3.78028	3.84297	3.91732	4.00233	4.09645
ε	2.4	2.6	2.8	3.0	3.5	4.0	4.5
T	4.05104	4.17446	4.30075	4.42955	4.76056	5.10177	5.45085
b	4.19833	4.30664	4.42027	4.53834	4.84895	5.17574	5.51419
ε	5.0	5.5	6.0	7.0	8.0	9.0	10.0
T	5.80616	6.16643	6.53093	7.26988	8.01912	8.77614	9.53918
b	5.86138	6.21529	6.57471	7.30600	8.04979	8.80277	9.56383

On the basis of these, the remaining characteristics of the oscillations can be determined by integrating the Cauchy problem. The data indicate that, when ε increases, the period and initial value of the velocity in normalized time τ, corresponding to the limit cycle, increase monotonically with a derivative with respect to ε (for $\varepsilon \geqslant 2$) of the order of unity. Moreover, it can be established that the ratio $b/T \simeq 1$ when $\varepsilon \gg 1$, i.e. $y/T \simeq 1$ for $t = 0$ and $\varepsilon \gg 1$. A comparison of the numerical results with analytical calculations using perturbation theory (the Lyapunov–Poincaré and Krylov–Bogolyubov methods) shows that the solution, to a first approximation in ε, approximates the numerical solution fairly well when $\varepsilon \sim 0.1$.

In Fig. 1 the solid curves represent the half-period of the oscillations T and the indicated value of the velocity b as a function of the self-excitation factor ε, $0 \leqslant \varepsilon \leqslant 10$. The algorithm described enables us to carry out accurate calculations for considerably larger values of $\varepsilon \sim 10^2 - 10^3$.

The last of the functions $x(\tau)$ and $y(\tau)$ in an interval equal to full period $0 \leqslant \tau \leqslant 2$ ($0 \leqslant t \leqslant 2T$ in the initial time), for characteristic values of ε are shown in Fig. 2. As was pointed out above, these functions satisfy the condition $x(\tau - 1) \equiv -x(\tau)$, $y(\tau - 1) \equiv -y(\tau)$; hence, we can confine ourselves to the interval $0 \leqslant \tau \leqslant 1$, i.e. $0 \leqslant t \leqslant T$. When $\varepsilon \leqslant 1$ the oscillations $x(\tau)$ are "close" to harmonic (see the

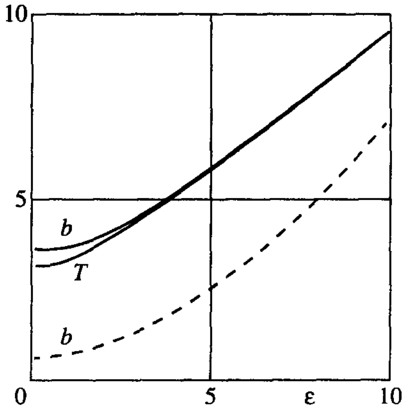


Fig. 1

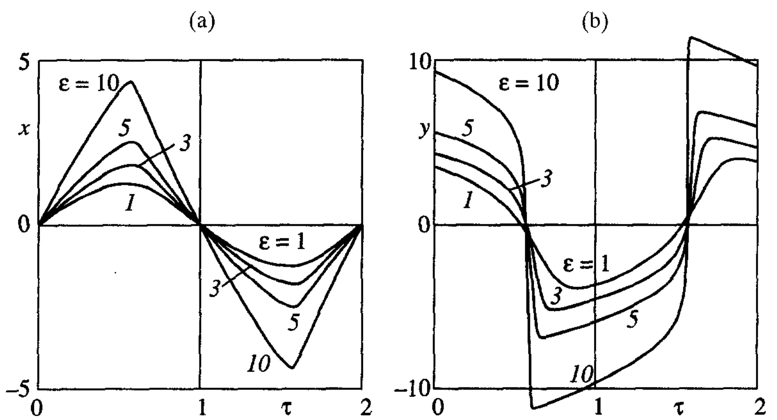


Fig. 2

curve for $\varepsilon = 1$ in Fig. 2a). When $\varepsilon > 1$ considerable deviations are observed, particularly the velocity $y(\tau)$ (see Fig. 2b), which differs from cosinusoidal. Starting with $\varepsilon = 2$ a relaxation form of oscillations is observed (with respect to the variable $y(\tau)$), which becomes very pronounced when $\varepsilon \gtrsim 5$ (see Fig. 2b).

These properties of the self-sustained oscillations show up quite clearly in the graphs of Fig. 3, which show the limit cycles in the phase plane (x, y) for different values of ε. We take the parameter $\tau = t/T$, $0 \leqslant \tau \leqslant 2$, related to the "natural" period of the oscillations, as the "natural" argument. When $\varepsilon \leqslant 1$ the limit cycles are "close" to an ellipse with semi-axes $(b/T, b)$, $b = 2\pi/\sqrt{3}$. As $\varepsilon(\varepsilon \gtrsim 5)$ increases in the second and fourth quadrants, sharp turns (of the corner-point type) of the tangents to the curve are observed (a large local curvature), connected with the practically instantaneous change of the variable $y(\tau)$ (see Fig. 2b).

Note that there are no detailed numerical-analytical investigations of the self-sustained oscillations of a Rayleigh oscillator for moderately large self-excitation factors in the available scientific literature, though there are sporadic numerical results [18, etc.]. The Rayleigh equations are the basis for describing many physical processes, whereas the Van der Pol equation, to which a considerable number of papers are devoted (see the list of references), is a corollary, obtained for quite burdensome additional conditions.

3. SELF-SUSTAINED OSCILLATIONS OF A VAN DER POL OSCILLATOR FOR MODERATELY LARGE SELF-EXCITATION FACTORS

Similar to the investigations presented in Section 2, we constructed the characteristics of self-sustained oscillations for the classical Van der Pol equation for moderately large values of ε

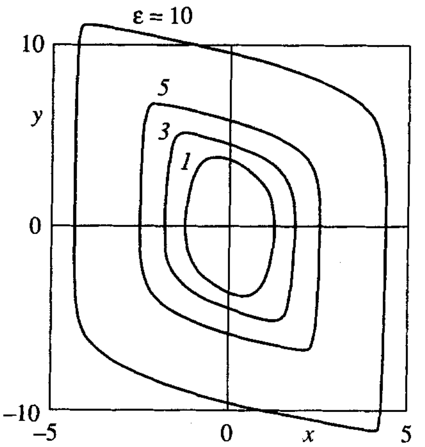


Fig. 3

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0, \quad 0 \le \varepsilon \le \varepsilon^* \quad (\varepsilon \sim 10)$$
$$x(0) = x(2T), \quad \dot{x}(0) = \dot{x}(2T) \tag{3.1}$$

This equation is obtained by differentiating Eq. (2.1) with respect to t and making the changes of variables $\sqrt{3}\dot{x} \to x$, $\sqrt{3}\ddot{x} \to \dot{x}$. Hence, the limit cycle $(x(t), \dot{x}(t))$ – a trajectory in the phase plane – for Eq. (3.1) is equivalent to the curve $\sqrt{3}(\dot{x}(t), \ddot{x}(t))$ in problem (2.1). This leads to an additional considerable instability in the calculations for large ε, in particular $\varepsilon \sim 10$ (beginning with $\varepsilon \approx 5$).

A unique stable periodic solution is constructed using the Lyapunov–Poincaré method [10]. The procedure of continuation with respect to the parameter ε and the method of accelerated convergence are then used in the same way as previously [16]. The boundary-value problem for the variables (x, y) and the sensitivity functions (p, q) and (w, z) with respect to the parameters b and T of the type (2.4) is reduced in normalized time (with argument) τ to the form

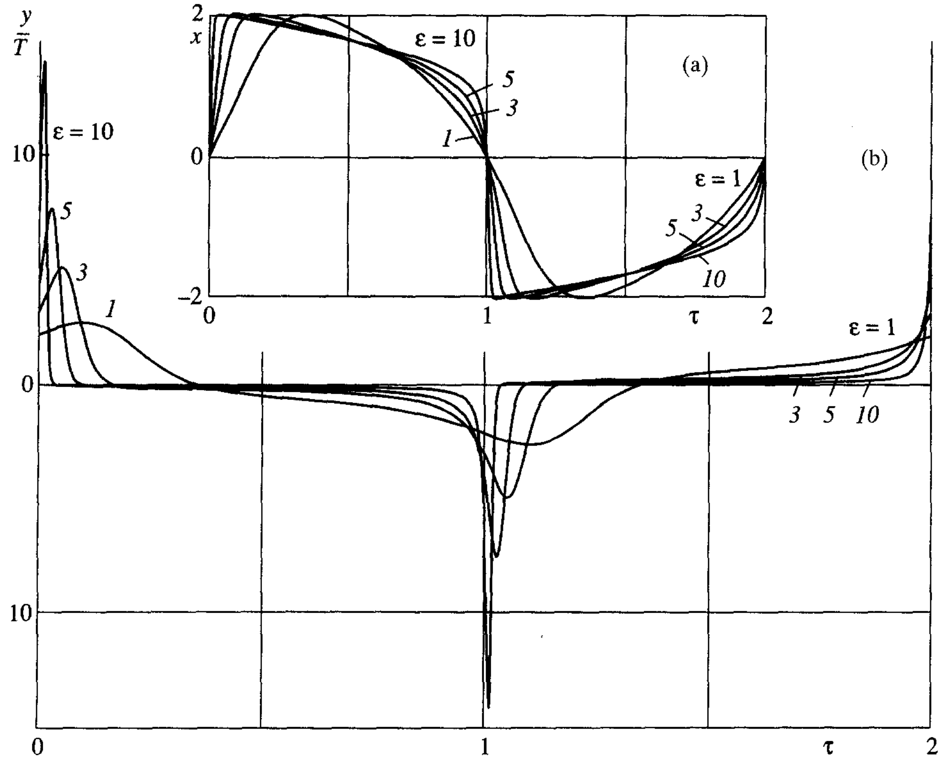
$$\begin{aligned}
&\dot{x} = y, \quad \dot{y} = -T^2x + \varepsilon T(1 - x^2)y, \quad x(0) = x(1) = 0, \quad y(0) = b, \quad y(1) = -b \\
&\dot{p} = q, \quad \dot{q} = -T^2p + \varepsilon T(1 - x^2)q - 2\varepsilon Txyp, \quad p(0) = 0, \quad q(0) = 1 \\
&\dot{w} = z, \quad \dot{z} = -T^2w + \varepsilon(1 - x^2)(y + Tz) - 2\varepsilon Txyw - 2Tx, \quad w(0) = z(0) = 0
\end{aligned} \tag{3.2}$$

The algorithm for solving problem (3.2) is similar to that described above. Refining corrections δT and δb are found recurrently from a system of the type (2.5), in which the coefficients and right-hand sides are obtained by integrating system (3.2) for specified T and b, beginning with $T_0 = T(0)$ and $b_0 = b(0)$, and the quantities $T(0)$ and $b(0)$ are found by the Lyapunov–Poincaré small-parameter method: $T(0) = \pi$ and $b(0) = 2\pi$. For the amplitude of the self-sustained oscillations $a(\varepsilon)$ we have $a_0 = a(0) = 2$. The values obtained for b_0 and a_0 are obviously considerably greater than for a Rayleigh oscillator (see Section 2). The half-periods $T(\varepsilon)$ for both problems must be identical. Calculations confirm that the determination of the function $T(\varepsilon)$, using scheme (3.2), (2.5), requires a greater number of iterations due to the above-mentioned instability for $\varepsilon \gg 1$, beginning with $\varepsilon \sim 5$.

If we use the highly accurate values for $T(\varepsilon)$ presented in Section 2, the calculations of the coefficient $b(\varepsilon)$ are simplified considerably and require the integration of system (3.2) for (x, y) and (p, q) for specified $T(\varepsilon)$.

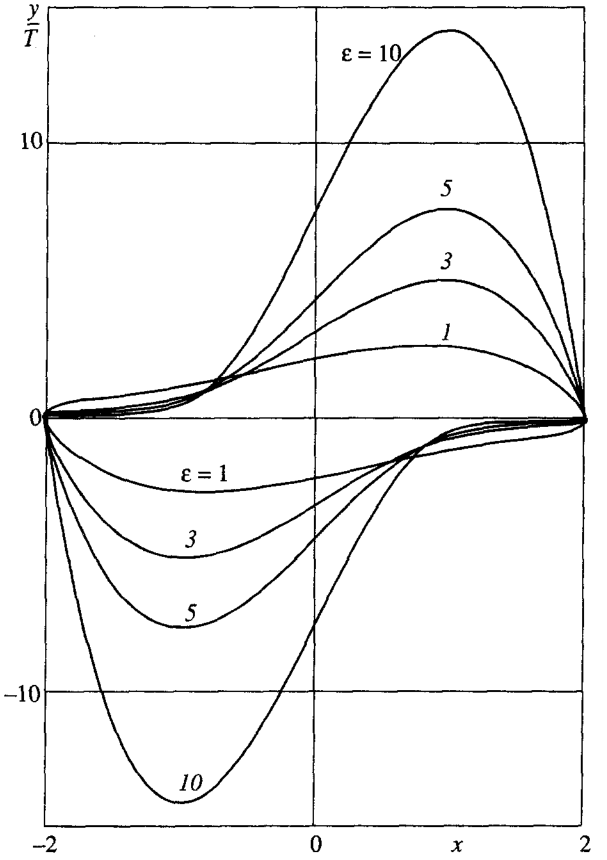
The results of calculations of y/T and $b(\varepsilon)$ using the general scheme are presented below:

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7
b	6.29386	6.32103	6.36886	6.43497	6.52101	6.62168	6.75056
ε	0.8	0.9	1.0	1.5	2.0	2.5	3.0
b	6.89422	7.05693	7.23847	8.41259	9.97483	11.86048	14.03403
ε	4.0	5.0	6.0	7.0	8.0	9.0	10.0
b	19.19177	25.39582	32.64521	40.93048	50.49040	60.62275	72.03648



y
T
10
0
10
ε = 10
5
3
1
(b)
ε = 1
3
5
10
0
1
τ
2
2
x
0
−2
ε = 10
5
3
1
(a)
ε = 1
3
5
10
0
1
τ
2

Fig. 4



y
T
10
0
−10
ε = 10
5
3
1
ε = 1
3
5
10
−2
0
x
2

Fig. 5

They confirm the above conclusions regarding the accuracy of the calculations. The function $b(\varepsilon)$ is presented in Fig. 1 on a reduced scale of 1:10 by the dashed curve. The graph of the function $T(\varepsilon)$ practically coincides with the curve corresponding to self-sustained oscillations of a Rayleigh oscillator.

In Fig. 4 we show graphs of the functions $x(\tau)$ and $y(\tau)$ for characteristic values of ε (graph b). The shapes of the curves $x(\tau)$ in Fig. 4(a) and $y(\tau)$ in Fig. 2(b) are identical (the curves are distinguished by a constant shift with respect to τ and by the scale). Important properties of the self-sustained oscillations are illustrated by the family of limit cycles (Fig. 5). The basic known property is the fact that $a(\varepsilon) \simeq 2$ ($a(0) = 2$) for all $\varepsilon > 0$ [11–16].

The main results of a numerical investigation of the self-sustained oscillations of a Van der Pol oscillator for moderately large values of $\varepsilon \sim 10$ and 20 can be found in papers published a long time ago [14, 15, etc.]. A comparison with these enables us to draw the following conclusions.

1. The method and algorithm described above are extremely effective and simple.
2. For each value of ε the algorithm enables the accuracy to be monitored. There are constructive methods for considerably increasing the accuracy. This property hardly exists in [14, 15] and it is difficult to monitor the accuracy (only two or three decimal places are reliable).
3. It is interesting to note that the values obtained above lie, as a rule, between the values given in [14] and in [15].

The proposed algorithms of the accelerated-convergence method and the method of continuation with respect to a parameter – the feedback factor – are extremely effective for finding periodic solutions and the limit cycles of systems described by the Rayleigh and Van der Pol type equations. They are based on the use of the properties of symmetry. The numerical-analytical investigations described enabled us to distinguish a number of characteristic features of the transition of systems from almost harmonic self-sustained oscillation modes to essentially non-linear (multifrequency) relaxation oscillations and to compare Rayleigh and Van der Pol oscillators for small and moderately large values of the feedback factors.

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